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Approximation by Boolean Sums of Positive Linear Operators. II. Gopengauz-Type Estimates

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1. INTRODUCTION

Let $\mathbb{N}=\{1, 2, ...\}$ be the set of natural numbers. For $f \in C[a, b]$ (real-valued and continuous functions on the compact interval $[a, b]$), let $\|f\| := \max\{|f(t)| : a \leqslant t \leqslant b\}$ denote the Čebyšev norm of f. Furthermore, let Π_n be the set of real algebraic polynomials of degree $\leqslant n$. By c, $\tilde{c}$ we denote positive absolute constants independent of n, f, and $x \in [a, b]$. The constants c and $\tilde{c}$ may be different at different occurrences even on the same line.

For $f \in C[a, b]$, the second order modulus of continuity $\omega_2(f, \delta)$ is defined by $(0 \leqslant \delta \leqslant \frac{1}{2}(b-a))$

$$\omega_2(f, \delta) := \sup\{|f(x-h) - 2f(x) + f(x+h)|, x, x \pm h \in [a, b], 0 \leqslant h \leqslant \delta\}.$$

In [10, 11] Dzjadyk and Freud proved the following

THEOREM A. *For $f \in C[-1, 1]$, $n \geqslant 2$, there exists a $p_n(f, \cdot) \in \Pi_n$ such that*

$$|f(x) - p_n(f, x)| \leqslant c \cdot \omega_2(f, \sqrt{1-x^2} \cdot n^{-1} + n^{-2}), \qquad |x| \leqslant 1. \quad (1.1)$$

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Defining $\Delta_n(x) := \max\{\sqrt{1-x^2}\cdot n^{-1}, n^{-2}\}$, we have

$$\Delta_n(x) \leqslant \sqrt{1-x^2}\cdot n^{-1} + n^{-2} \leqslant 2\cdot \Delta_n(x). \tag{1.2}$$

From (1.1) we arrive at

$$|f(x) - p_n(f, x)| \leqslant c\cdot \omega_2(f, \Delta_n(x)), \qquad |x| \leqslant 1. \tag{1.3}$$

In [17] Gopengauz proved

THEOREM B. *For $f \in C[-1, 1]$, $n \geqslant 2$, there exists a $p_n(f, \cdot) \in \Pi_n$ such that*

$$|f(x) - p_n(f, x)| \leqslant c\cdot \omega_2(f, \sqrt{1-x^2}\cdot n^{-1}), \qquad |x| \leqslant 1. \tag{1.4}$$

This result was also obtained by DeVore [8, Theorem 3].

We note, for the sake of completeness, that in a series of recent papers, a problem posed by Lorentz and Stečkin, namely that of replacing $\Delta_n(x)$ by the quantity $\sqrt{1-x^2}\cdot n^{-1}$ in the more general inequalities of the type

$$|f^{(k)}(x) - p_n^{(k)}(f, x)| \leqslant c\cdot \Delta_n(x)^{r-k}\cdot \omega_s(f^{(r)}, \Delta_n(x)),$$

$r, s = 0, 1, 2, ...$, $f \in C^r[-1, 1]$, $0 \leqslant k \leqslant r$, was completely solved. It was shown, among other things, that Gopengauz' original conjecture, namely, the possibility of such a replacement, for $0 \leqslant k \leqslant r$, is not true in general. See [6, 16, 24] for details.

The aim of the present note is to show that the Gopengauz-type estimate (1.4) involving the second order modulus of smoothness ω_2 can be obtained using rather simple modifications of certain sequences of positive linear operators $G_{m(n)}$. These will be introduced in the next paragraph.

In [18, 22] Pičugov and Lehnhoff constructed the following operators $G_{m(n)}$.

Let $f \in C[-1, 1]$, $K_{m(n)}(v) := \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cos kv$. Then for $n \in \mathbb{N}$

$$G_{m(n)}(f, x) := \pi^{-1} \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{m(n)}(v)\, dv. \tag{1.5}$$

Here the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ with (i) $K_{m(n)}$ positive and even, and (ii) $\int_{-\pi}^{\pi} K_{m(n)}(v)\, dv = \pi$. This implies that $G_{m(n)}(f, \cdot)$ is an algebraic polynomial of degree $m(n)$.

For $s \in \mathbb{N}$ Matsuoka [21] (see also [7, p. 79 ff.]) investigated the following special kernels,

$$K_{sn-s}(v) = c_{n,s}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s},$$

where $c_{n,s}$ is chosen such that $\pi^{-1}\int_{-\pi}^{\pi} K_{sn-s}(v)\,dv=1$. From (1.5) one obtains corresponding operators G_{sn-s} being based upon the kernels K_{sn-s}.

Pičugov and Lehnhoff published estimates involving the least concave majorant of the first order modulus ω_1 and the first order modulus itself. For instance, it was shown by Lehnhoff [18] that one has

$$|G_{3n-3}(f,x)-f(x)|\leqslant 4\cdot(\omega_1(f,\sqrt{1-x^2}\cdot n^{-1})+\omega_1(f,|x|\cdot n^{-2}))$$

for all $f\in C[-1,1]$ and $|x|\leqslant 1$.

The investigation of both authors mentioned was supplemented and extended in several papers by Lehnhoff [19] and the present authors (see [2–5, 13–15]).

An important tool used in all papers mentioned is the Boolean sum of the operators $G_{m(n)}$ and certain interpolation operators L. In particular, it was conjectured in the second author's paper [14] that for a certain modification G^1_{3n-3} (to be defined below) of the operators G_{3n-3} the following Gopengauz-type inequality holds:

Conjecture. Let $n\geqslant 2$ and $f\in C[-1,1]$. Then

$$|G^1_{3n-3}(f,x)-f(x)|\leqslant c\cdot\omega_2(f,\sqrt{1-x^2}\cdot n^{-1}),\qquad |x|\leqslant 1.$$

In the present paper we shall show that this is indeed the case, and that certain simpler operators G^+_{3n-3} have the same property. Our results are obtained via the use of more general assertions which may be of interest in themselves. We establish a general theorem (Theorem 5.2), prove Gonska's conjecture, and we show that his conjecture also holds for the more general operators $G^+_{sn-s}=G^1_{sn-s}$, $s\geqslant 3$ (Theorem 5.5).

2. Notes on the Boolean Sum Method

An aspect returning in all papers just mentioned is the use of the so-called Boolean sum $A\oplus B$ of certain linear operators A and B. This mapping is defined by the equality $A\oplus B:=A+B-A\circ B$ (subject to suitable domains and ranges of A and B). To be more specific, let Lf denote the linear function interpolating f at -1 and 1, i.e.,

$$L(f,x)=\tfrac{1}{2}f(1)(x+1)+\tfrac{1}{2}f(-1)(1-x).$$

In his paper [19] Lehnhoff used operators of the type $G^+_{m(n)}:=L\oplus G_{m(n)}$ to arrive at a Teljakowskiĭ-type inequality.

Operators of the symmetric form $G^*_{m(n)} := G_{m(n)} \oplus L$ were considered from a more general point of view in [12]. See [5] for further results and additional references.

The natural "successors" $G^1_{m(n)} := L \oplus G_{m(n)} \oplus L$ (note that "$\oplus$" is an associative, but in general a non-commutative operation) were investigated in [14]. It turns out in Corollary 2.2 below, however, that in the special situation under consideration here, we have $G^1_{m(n)} = G^+_{m(n)}$.

The consideration of the three types of Boolean sum operators just listed (which were implicitly also used in DeVore's paper [8]) is motivated by the following variant of a theorem by Barnhill and Gregory [1].

THEOREM 2.1. *Let P and Q be linear operators mapping a function space G (consisting of functions on the domain D) into a subspace H of G. Let G_0 be a subset of G, and let $\mathscr{L} = \{l\}$ be a set of linear functionals defined on H.*

(i) *Let $l(Pf) = l(f)$ for all $l \in \mathscr{L}$ and all $f \in H$. Then $l((P \oplus Q)f) = lf$ for all $l \in \mathscr{L}$ and all $f \in H$.*

(ii) *Let $Qf = f$ for all $f \in G_0$. Then $(P \oplus Q)f = f$ for all $f \in G_0$.*

(iii) *Let f and Qf be in the set of all functions g such that $Pg = g$. Then $(P \oplus Q)f = f$.*

In other words, $P \oplus Q$ inherits certain "interpolation properties" of P, the function precision of Q, and also some function precision properties of P.

Proof. (i) Let $l \in \mathscr{L}$ and $f \in H$. Then

$$\begin{aligned} l((P \oplus Q)f) &= l(Pf) + l(Qf) - l(PQf) \\ &= l(f) + l(Qf) - l(Qf) \qquad \text{since} \quad Qf \in H \\ &= l(f). \end{aligned}$$

(ii) For $f \in G_0$ there holds

$$(P \oplus Q)f = Pf + Qf - PQf = Pf + f - Pf = f.$$

(iii) From the assumption it follows that for the function f in question there holds $P(Qf) = Qf$. Hence

$$(P \oplus Q)f = Pf + Qf - PQf = f + Qf - Qf = f. \quad \blacksquare$$

For the operators at hand, namely L and $G_{m(n)}$, we have the following

COROLLARY 2.2. *The operator $G^+_{m(n)} = L \oplus G_{m(n)}$ has the following properties:*

(i) $G^+_{m(n)}(f; \pm 1) = f(\pm 1)$ *for all* $f \in C[-1, 1]$.

(ii) $G^+_{m(n)} f = f$ *for all* $f \in \Pi_1$.

(iii) $G^+_{m(n)} = G^1_{m(n)}$ $(= L \oplus G_{m(n)} \oplus L)$.

Proof. (i) Follows from the interpolation properties of L at -1 and $+1$.

(ii) For $f \in \Pi_1$ we have $Lf = f$ and $G_{m(n)} f \in \Pi_1$. The latter statement is a consequence of the equalities $G_{m(n)}(1, x) = 1$ and $G_{m(n)}(t, x) = \rho_{1,m(n)} \cdot x$ (see [14]). Theorem 2.1(iii) then implies $G^+_{m(n)} f = f$.

(iii) For any $f \in C[-1, 1]$ there holds

$$\begin{aligned} G^1_{m(n)} f &= (L \oplus G_{m(n)} + L - (L \oplus G_{m(n)}) \circ L)(f) \\ &= (L \oplus G_{m(n)})(f) + L(f) - (L \oplus G_{m(n)})(Lf). \end{aligned}$$

Since Lf is a linear function we have by (ii) that $G^+_{m(n)}(Lf) = (L \oplus G_{m(n)})(Lf) = Lf$, implying $G^1_{m(n)} f = G^+_{m(n)} f$. ∎

3. A Jackson-Type Inequality for certain Boolean Sum Operators

Let A_n be a sequence of positive linear operators mapping $C[-1, 1]$ into $C[-1, 1]$. We consider the sequence of operators $A_n^+ := L \oplus A_n$ where L is given as above. Hence for $f \in C[-1, 1]$ and $|x| \leqslant 1$ we have

$$\begin{aligned} A_n^+(f, x) = A_n(f, x) + \{&\tfrac{1}{2} \cdot (x+1) \cdot [f(1) - A_n(f, 1)] \\ &+ \tfrac{1}{2} \cdot (1 - x) \cdot [f(-1) - A_n(f, -1)]\}. \end{aligned}$$

In the following $C^2[a, b]$ denotes the space of twice continuously differentiable functions.

Lemma 3.1. *Let* $n \in \mathbb{N}$ *and let* A_n: $C[-1, 1] \to C[-1, 1]$ *be a sequence of positive linear operators, satisfying the following conditions*:

(i) $A_n(1, x) = 1$,

(ii) $A_n(t, x) = \lambda_n x$, $1 - \lambda_n = O(n^{-2})$,

(iii) $A_n((t-x)^2, x) = O((1 - x^2) \cdot n^{-2} + n^{-4})$,

where O *is the Landau symbol. Then for* $h \in C^2[-1, 1]$ *and* $|x| \leqslant 1$ *the following inequality holds*:

$$|A_n^+(h, x) - h(x)| \leqslant c \cdot ((1 - x^2) \cdot n^{-2} + n^{-4}) \cdot \|h''\|.$$

Proof. If $|x| \leqslant 1$ and $h \in C^2[-1, 1]$, using Taylor's formula we know that there exists a ξ between t and x such that

$$h(t) - h(x) - h'(x)(t-x) = \tfrac{1}{2}(t-x)^2 h''(\xi),$$

where, if $x = 1$, then $h'(1) := h'_-(1)$, and if $x = -1$, then $h'(-1) := h'_+(-1)$. This gives the estimate

$$|h(t) - h(x) - h'(x)(t-x)| \leqslant \tfrac{1}{2}(t-x)^2 \|h''\|.$$

Since $A_n(1, x) = 1$ and A_n is a sequence of positive operators, we have

$$|A_n(h, x) - h(x) - h'(x) \cdot A_n(t-x, x)| \leqslant \tfrac{1}{2} A_n((t-x)^2, x) \cdot \|h''\|, \quad (3.1)$$

and

$$A_n(t-x, x) = A_n(t, x) - x \cdot A_n(1, x) = (\lambda_n - 1)x; \quad (3.2)$$

hence

$$|A_n(h, x) - h(x) - h'(x)(\lambda_n - 1)x| \leqslant \tfrac{1}{2} A_n((t-x)^2, x) \cdot \|h''\|. \quad (3.3)$$

Letting $x = 1$ in (3.3) we have

$$|A_n(h, 1) - h(1) - h'(1)(\lambda_n - 1)| \leqslant \tfrac{1}{2} A_n((t-1)^2, 1) \cdot \|h''\|.$$

From condition (iii) we know that $A_n((t-1)^2, 1) = O(n^{-4})$, hence

$$\begin{aligned} &|\tfrac{1}{2}(x+1)[A_n(h, 1) - h(1)] - \tfrac{1}{2}(x+1) h'(1)(\lambda_n - 1)| \\ &\quad \leqslant \tfrac{1}{4}(x+1) A_n((t-1)^2, 1) \cdot \|h''\| \\ &\quad \leqslant \tfrac{1}{2} A_n((t-1)^2, 1) \cdot \|h''\| \\ &\quad = O(n^{-4}) \cdot \|h''\|. \end{aligned} \quad (3.4)$$

In (3.3) letting $x = -1$ we have

$$|A_n(h, -1) - h(-1) - h'(-1)(1 - \lambda_n)| \leqslant \tfrac{1}{2} A_n((t+1)^2, -1) \cdot \|h''\|.$$

Because of $A_n((t+1)^2, -1) = O(n^{-4})$, we arrive at

$$\begin{aligned} &|\tfrac{1}{2}(1-x)[A_n(h, -1) - h(-1)] - \tfrac{1}{2}(1-x) h'(-1)(1 - \lambda_n)| \\ &\quad \leqslant \tfrac{1}{4}(1-x) A_n((t+1)^2, -1) \cdot \|h''\| \\ &\quad \leqslant \tfrac{1}{2} A_n((t+1)^2, -1) \cdot \|h''\| \\ &\quad = O(n^{-4}) \cdot \|h''\|. \end{aligned} \quad (3.5)$$

Now we define

$$e_n(x) := \tfrac{1}{2}(x+1)[A_n(h, 1) - h(1)] + \tfrac{1}{2}(1-x)[A_n(h, -1) - h(-1)],$$
$$d_n(x) := \tfrac{1}{2}(x+1)\, h'(1)(\lambda_n - 1) + \tfrac{1}{2}(1-x)\, h'(-1)(1-\lambda_n).$$

From (3.4) and (3.5) it follows that

$$|e_n(x) - d_n(x)| \leqslant O(n^{-4}) \cdot \|h''\|, \tag{3.6}$$

and from the definition of $A_n^+(h, x)$ we get

$$A_n^+(h, x) = A_n(h, x) - e_n(x)$$

and

$$\begin{aligned} A_n^+(h, x) - h(x) &= A_n(h, x) - h(x) - h'(x)\, x(\lambda_n - 1) \\ &\quad + h'(x)\, x(\lambda_n - 1) - e_n(x) + d_n(x) - d_n(x) \\ &= [A_n(h, x) - h(x) - h'(x)\, x(\lambda_n - 1)] \\ &\quad - [e_n(x) - d_n(x)] + [h'(x)\, x(\lambda_n - 1) - d_n(x)]. \end{aligned}$$

From (3.3), (3.6), and condition (iii) we obtain

$$\begin{aligned} |A_n^+(h, x) - h(x)| &\leqslant |A_n(h, x) - h(x) - h'(x)\, x(\lambda_n - 1)| \\ &\quad + |e_n(x) - d_n(x)| + |h'(x)\, x(\lambda_n - 1) - d_n(x)| \\ &\leqslant \tfrac{1}{2} A_n((t-x)^2, x) \cdot \|h''\| + O(n^{-4}) \cdot \|h''\| \\ &\quad + |h'(x)\, x(\lambda_n - 1) - d_n(x)| \\ &= O((1-x^2)n^{-2} + n^{-4}) \cdot \|h''\| + I_n(x), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} I_n(x) &:= |h'(x)\, x(\lambda_n - 1) - d_n(x)| \\ &= |h'(x)\, x(\lambda_n - 1) - \tfrac{1}{2}(x+1)\, h'(1)(\lambda_n - 1) + \tfrac{1}{2}(1-x)\, h'(-1)(\lambda_n - 1)| \\ &= |\lambda_n - 1| \cdot |h'(x)x - \tfrac{1}{2}(x+1)\, h'(1) + \tfrac{1}{2}(1-x)\, h'(-1)|. \end{aligned}$$

Since $x = \tfrac{1}{2}(x+1) - \tfrac{1}{2}(1-x)$, we can write

$$\begin{aligned} I_n(x) &= |1 - \lambda_n| \cdot |\tfrac{1}{2}(x+1)[h'(x) - h'(1)] + \tfrac{1}{2}(1-x)[h'(-1) - h'(x)]| \\ &\leqslant |1 - \lambda_n| \cdot \{\tfrac{1}{2}(x+1)|h'(x) - h'(1)| + \tfrac{1}{2}(1-x)|h'(-1) - h'(x)|\}. \end{aligned}$$

Using the mean value theorem we have

$$I_n(x) \leqslant |1-\lambda_n| \cdot \{\tfrac{1}{2}(x+1)|h''(\theta)| \cdot |1-x| + \tfrac{1}{2}(1-x)|h''(\eta)| \cdot |x+1|\},$$

where $-1<\theta<1$ and $-1<\eta<1$, hence

$$\begin{aligned} I_n(x) &\leqslant |1-\lambda_n| \cdot \{\tfrac{1}{2}(1-x^2)+\tfrac{1}{2}(1-x^2)\} \cdot \|h''\| \\ &= |1-\lambda_n| \cdot (1-x^2) \cdot \|h''\|. \end{aligned}$$

From condition (ii) we have

$$I_n(x) \leqslant c \cdot (1-x^2) \cdot n^{-2} \cdot \|h''\|, \tag{3.8}$$

and from (3.7) and (3.8) we derive that

$$\begin{aligned} &|A_n^+(h, x) - h(x)| \\ &\quad \leqslant \{c \cdot ((1-x^2) \cdot n^{-2} + n^{-4}) + c \cdot (1-x^2) \cdot n^{-2}\} \cdot \|h''\| \\ &\quad \leqslant c \cdot ((1-x^2) \cdot n^{-2} + n^{-4}) \cdot \|h''\|. \quad \blacksquare \end{aligned}$$

Remark 3.2. The inequality of Lemma 3.1 implies that $A_n^+ = L \oplus A_n$ reproduces linear functions. This follows also from Theorem 2.1(iii).

4. Further Auxiliary Results

Lemma 4.1. *Let $m(n) \in \mathbb{N}$ and $c \cdot n \leqslant m(n) \leqslant \tilde{c} \cdot n$. Furthermore, let $p_{m(n)} \in \Pi_{m(n)}$ and let ω be a modulus of continuity (i.e., $\omega(h) \to 0$ for $h \to 0$, ω is positive and increasing, and ω is subadditive). If*

$$|p_{m(n)}(x)| \leqslant \Delta_n(x) \cdot \omega(\Delta_n(x)), \qquad |x| \leqslant 1,$$

then

$$|p'_{m(n)}(x)| \leqslant c \cdot \omega(\Delta_n(x)), \qquad |x| \leqslant 1.$$

Proof. The proof is similar to that of Theorem 3 in [20, p. 71].

Lemma 4.2. *If $n \geqslant 2$ and $h \in C^2[-1, 1]$, then there exists a polynomial $A_n(h, \cdot) \in \Pi_n$ such that for $|x| \leqslant 1$ one has*

(i) $|h(x) - A_n(h, x)| \leqslant c \cdot \Delta_n^2(x) \cdot \|h''\|$, *and*

(ii) $|h'(x) - A_n'(h, x)| \leqslant c \cdot \Delta_n(x) \cdot \|h''\|$, *where* $A_n'(h, x) := (d/dx) A_n(h, x)$.

Proof. See Trigub [23, Lemma 1].

LEMMA 4.3. *Let* $n \geqslant 2$, $m(n) \in \mathbb{N}$, *and* $c \cdot n \leqslant m(n) \leqslant \tilde{c} \cdot n$. *Let* A_n: $C[-1, 1] \to \Pi_{m(n)}$ *be a sequence of positive linear algebraic polynomial operators, satisfying conditions* (i)–(iii) *of Lemma* 3.1. *If* $h \in C^2[-1, 1]$, *then*

$$\left|\frac{d}{dx} A_n^+(h, x) - h'(x)\right| \leqslant c \cdot \Delta_n(x) \cdot \|h''\|, \qquad |x| \leqslant 1.$$

Proof. Note that $\Delta_n^2(x) = \max\{(1-x^2)n^{-2}, n^{-4}\}$. Writing $W_n(h, x) := A_n^+(h, x)$, we get from Lemma 3.1 that

$$|W_n(h, x) - h(x)| \leqslant c \cdot ((1-x^2) \cdot n^{-2} + n^{-4}) \cdot \|h''\| \tag{4.1}$$

$$\leqslant c \cdot \Delta_n^2(x) \cdot \|h''\|. \tag{4.2}$$

Since $n \geqslant 2$, with $\Lambda_n(h, \cdot)$ as in Lemma 4.2, we have for $|x| \leqslant 1$

$$|h(x) - \Lambda_n(h, x)| \leqslant c \cdot \Delta_n^2(x) \cdot \|h''\|, \tag{4.3}$$

and

$$|h'(x) - \Lambda_n'(h, x)| \leqslant c \cdot \Delta_n(x) \cdot \|h''\|. \tag{4.4}$$

Thus

$$|W_n(h, x) - \Lambda_n(h, x)| \leqslant |W_n(h, x) - h(x)| + |h(x) - \Lambda_n(h, x)|$$

$$\leqslant c \cdot \Delta_n^2(x) \cdot \|h''\|. \tag{4.5}$$

The degree of $W_n(h, \cdot) - \Lambda_n(h, \cdot)$ is $m'(n) = \max\{m(n), n\}$. Since $c \cdot n \leqslant m(n) \leqslant \tilde{c} \cdot n$, the same is true for $m'(n)$. Applying Lemma 4.1 (with $\omega(t) = c \cdot \|h''\| \cdot t$, where c is the constant from (4.5)) we arrive at

$$|W_n'(h, x) - \Lambda_n'(h, x)| \leqslant c \cdot \Delta_n(x) \cdot \|h''\|. \tag{4.6}$$

From (4.6) and (4.4) it follows that

$$|W_n'(h, x) - h'(x)| \leqslant c \cdot \Delta_n(x) \cdot \|h''\|, \qquad |x| \leqslant 1,$$

which yields the claim of Lemma 4.3. ∎

5. GOPENGAUZ-TYPE INEQUALITIES

This section contains the main result of our paper (Theorem 5.2). Its proof is obtained by the smoothing method which is described in the following

LEMMA 5.1. *Let* $H_n: C[-1, 1] \to C[-1, 1]$ *be a sequence of linear operators, satisfying the following conditions*:

(i) $\|H_n f\| \leqslant c \cdot \|f\|$ *for all* $f \in C[-1, 1]$.

(ii) *There is a function* $\varepsilon_n: [-1, 1] \to [0, 1]$ *such that for all* $g \in C^2[-1, 1]$ *there holds*

$$|H_n(g, x) - g(x)| \leqslant c \cdot \varepsilon_n^2(x) \cdot \|g''\|, \qquad |x| \leqslant 1.$$

Then we have for all $f \in C[-1, 1]$

$$|H_n(f, x) - f(x)| \leqslant c \cdot \omega_2(f, \varepsilon_n(x)), \qquad |x| \leqslant 1.$$

Proof. Lemma 5.1 is obtained by using the K-functional method (see, e.g., DeVore [9]). ∎

THEOREM 5.2. *Let* $n \geqslant 2$, $m(n) \in \mathbb{N}$, *and* $c \cdot n \leqslant m(n) \leqslant \tilde{c} \cdot n$. *Furthermore, let* $A_n: C[-1, 1] \to \Pi_{m(n)}$ *be a sequence of positive linear operators, satisfying conditions* (i)–(iii) *of Lemma* 3.1. *Then we have for all* $f \in C[-1, 1]$ *and all* $|x| \leqslant 1$ *that*

$$|A_n^+(f, x) - f(x)| \leqslant c \cdot \omega_2(f, \sqrt{1 - x^2} \cdot n^{-1}).$$

Proof. We have to show that for the operators A_n^+ the conditions (i) and (ii) of Lemma 5.1 hold with $\varepsilon_n(x) = \sqrt{1 - x^2} \cdot n^{-2}$.

We first show that (ii) is satisfied. To this end we define again $W_n(g, x) := A_n^+(g, x)$. For any $g \in C^2[-1, 1]$ we know from (4.2) that

$$|g(x) - W_n(g, x)| \leqslant c \cdot \Delta_n^2(x) \cdot \|g''\|. \tag{5.1}$$

Inequality (5.1) can be improved near the endpoints by using the fact that $W_n(g, \pm 1) = g(\pm 1)$. For example, in the case $0 \leqslant x \leqslant 1$ we arrive at

$$\begin{aligned} |g(x) - W_n(g, x)| &\leqslant |x - 1| \cdot |g'(\xi) - W_n'(g, \xi)| \\ &\leqslant c \cdot |x - 1| \cdot \Delta_n(\xi) \cdot \|g''\| \\ &\leqslant c \cdot (1 - x^2) \cdot \Delta_n(x) \cdot \|g''\|, \end{aligned} \tag{5.2}$$

where in the first inequality we used the mean value theorem with $x < \xi < 1$, in the second inequality we employed Lemma 4.3, and in the third inequality we made use of the fact that $1 - x \leqslant 1 - x^2$ for $0 \leqslant x \leqslant 1$ and $\Delta_n(\xi) \leqslant \Delta_n(x)$ (since $0 \leqslant x < \xi$). The same inequality as the one in (5.2) holds if $-1 \leqslant x \leqslant 0$. Hence we have

$$|g(x) - W_n(g, x)| \leqslant c \cdot (1 - x^2) \cdot \Delta_n(x) \cdot \|g''\|, \qquad |x| \leqslant 1. \tag{5.3}$$

Using a standard argument, (5.1) and (5.3) imply

$$|g(x)-A_n^+(g,x)|\leqslant c\cdot(1-x^2)\cdot n^{-2}\cdot\|g''\|,\qquad |x|\leqslant 1. \tag{5.4}$$

To verify condition (i) of Lemma 5.1, we note that the positivity of A_n implies for all $f\in C[-1,1]$ and $|x|\leqslant 1$ the inequality

$$|A_n(f,x)|\leqslant|A_n(1,x)|\cdot\|f\|=\|f\|.$$

Thus

$$\begin{aligned}|A_n^+(f,x)|&\leqslant|A_n(f,x)|+\tfrac{1}{2}(x+1)\cdot[|f(1)|+|A_n(f,1)|]\\&\quad+\tfrac{1}{2}(1-x)\cdot[|f(-1)|+|A_n(f,-1)|]\\&\leqslant\|f\|+(x+1)\cdot\|f\|+(1-x)\cdot\|f\|=3\|f\|,\end{aligned} \tag{5.5}$$

and from (5.5) and (5.4), using Lemma 5.1, we obtain Theorem 5.2. ∎

In the following we apply Theorem 5.2 to the operators $G_{m(n)}$.

LEMMA 5.3. *For $|x|\leqslant 1$ the following equality holds*

$$G_{m(n)}((t-x)^2,x)=\tfrac{1}{2}(1-\rho_{2,m(n)})(1-x^2)+\{3/2-2\rho_{1,m(n)}+\tfrac{1}{2}\rho_{2,m(n)}\}\,x^2.$$

Proof. See Lehnhoff [18].

THEOREM 5.4. *Let $n\geqslant 2$ and $c\cdot n\leqslant m(n)\leqslant\tilde{c}\cdot n$. Furthermore, let $K_{m(n)}(v)\geqslant 0$ and*

(i) $1-\rho_{1,m(n)}=O(n^{-2})$,

(ii) $\frac{3}{2}-2\rho_{1,m(n)}+\frac{1}{2}\rho_{2,m(n)}=O(n^{-4})$.

Then for all $f\in C[-1,1]$

$$|G_{m(n)}^+(f,x)-f(x)|\leqslant c\cdot\omega_2(f,\sqrt{1-x^2}\cdot n^{-1}),\qquad |x|\leqslant 1.$$

Proof. In [14] it was proved that

$$G_{m(n)}(1,x)=1\qquad\text{and}\qquad G_{m(n)}(t,x)=\rho_{1,m(n)}x.$$

Since $K_{m(n)}(v)\geqslant 0$ we have (see Cao and Gonska [5])

$$0<1-\rho_{2,m(n)}\leqslant 4\cdot(1-\rho_{1,m(n)})=O(n^{-2}).$$

From condition (ii) and Lemma 5.3 we obtain

$$G_{m(n)}((t-x)^2,x)=O((1-x^2)\cdot n^{-2}+n^{-4})$$

which, using Theorem 5.2, yields the claim of Theorem 5.4. ∎

THEOREM 5.5. *Let $n \geqslant 2$ and $s \geqslant 3$. Then for $f \in C[-1, 1]$ there holds*

$$|G^{+}_{sn-s}(f, x) - f(x)| \leqslant c \cdot \omega_2(f, \sqrt{1-x^2} \cdot n^{-1}), \qquad |x| \leqslant 1.$$

Proof. First observe that $n \leqslant sn - s \leqslant sn$ ($n \geqslant 2$ and $s \geqslant 2$) and that $K_{sn-s}(v) \geqslant 0$. It was proved in [7] that

$$1 - \rho_{1,sn-s} = O(n^{-2}), \qquad s \geqslant 2.$$

We also have (see Cao and Gonska [5])

$$\tfrac{3}{2} - 2\rho_{1,sn-s} + \tfrac{1}{2}\rho_{2,sn-s} = O(n^{-4}), \qquad s \geqslant 3.$$

Using Theorem 5.4 we obtain Theorem 5.5. ∎

Remark 5.6. In view of Corollary 2.2(iii) all estimates given above also hold for the corresponding operators $G^{1}_{m(n)}$. Thus Theorem 5.5 proves the conjecture of Cao and Gonska [5] (containing the second author's conjecture from [14] for the special case $s = 3$).

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